The Zero-Divisor Graph of a Commutative Semigroup: A Survey

David F. Anderson and Ayman Badawi

Abstract Let S be a (multiplicative) commutative semigroup with 0. Associate to S a (simple) graph $G(S)$ with vertices the nonzero zero-divisors of S, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this survey article, we collect some properties of the zero-divisor graph $G(S)$.

Keywords Zero-divisor graph • Semigroup • Poset • Lattice • Semi-lattice • Annihilator graph

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1 Introduction

Let R be a commutative ring with $1 \neq 0$, and let $Z(R)$ be its set of zero-divisors. Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between their corresponding ring-theoretic and graph-theoretic properties; for recent survey articles, see [13, 17, 18, 29, 56, 58], and [61]. For example, as in [11], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [23], who let all the elements of R be vertices and was mainly interested in colorings (also see [7]). The zero-divisor

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graph of a commutative ring R has been studied extensively by many authors. For other types of graphs associated to a commutative ring, see [2–4, 8–10, 16, 19–21, 24, 43, 55, 57, 59, 63, 67], and [73].

The concept of zero-divisor graph of a commutative ring in the sense of Anderson-Livingston as in [11] was extended to the zero-divisor graph of a commutative semigroup by DeMeyer, McKenzie, and Schneider in [33]. Let S be a (multiplicative) commutative semigroup with 0 (i.e., $0x = 0$ for every $x \in S$), and let $Z(S) = \{x \in S \mid xy = 0 \text{ for some } 0 \neq y \in S\}$ be the set of zero-divisors of S. As in [33], the *zero-divisor graph* of S is the (simple) graph $G(S)$ with vertices $Z(S) \setminus \{0\}$, the set of nonzero zero-divisors of S, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graph of a commutative semigroup with 0 has also been studied by many authors, for example, see [8, 9, 15, 30, 32–38, 41, 44, 46, 49, 51–54, 68, 69, 71], and [74–81].

The purpose of this survey article is to collect some properties of the zero-divisor graph of a commutative semigroup with 0. Our aim is to give the flavor of the subject, but not be exhaustive. In Sect. 2, we give several examples of zero-divisor graphs of semigroups. In Sect. 3, we give some properties of $G(S)$ and investigate which graphs can be realized as $G(S)$ for some commutative semigroup S with 0. In Sect. 4, we continue the investigation of which graphs can be realized as $G(S)$ and are particularly interested in the number (up to isomorphism) of such semigroups S. Finally, in Sect. 5, we briefly give some more results and references for further reading. An extensive bibliography is included.

Throughout, G will be a simple graph with $V(G)$ its set of vertices, i.e., G is undirected with no multiple edges or loops. We say that G is *connected* if there is a path between any two distinct vertices of G. For vertices x and y of G, define $\mathrm{d}(x, y)$ to be the length of a shortest path from x to y ($\mathrm{d}(x, x) = 0$ and $\mathrm{d}(x, y) = \infty$ if there is no path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles).

A graph G is *complete* if any two distinct vertices of G are adjacent. The complete graph with n vertices will be denoted by K_n (we allow n to be an infinite cardinal number). A *complete bipartite graph* is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices of G are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton, then we call G a *star graph*. We denote the complete bipartite graph by $K_{m,n}$, where $|A| = m$ and $|B| = n$ (again, we allow m and n to be infinite cardinals); so a star graph is a $K_{1,n}$.

Let H be a subgraph of a graph G. Then H is an *induced subgraph* of G if every edge in G with endpoints in H is also an edge in H, and G is a *refinement* of H if $V(H) = V(G)$. For a vertex x of a graph G, let $N(x)$ be the set of vertices in G that are adjacent to x and $\overline{N(x)} = N(x) \cup \{x\}$. A vertex x of G is called an *end* if there is only one vertex adjacent to x (i.e., if $|N(x)| = 1$). The *core* of G is the largest subgraph of G in which every edge is the edge of a cycle in G. Also, recall that a *component*, say C, of a graph G is a connected induced subgraph of G such that $a - b$ is not an edge of G for every vertex a of C and every vertex b of $G \setminus C$. It is known that every graph is a union of disjoint components.

Let S be a (multiplicative) commutative semigroup with 0. A $\emptyset \neq I \subseteq S$ is an *ideal* of S if $xI \subseteq I$ for every $x \in S$. A proper ideal I of S is a *prime ideal* if $xy \in I$ for $x, y \in S$ implies $x \in I$ or $y \in I$. An $x \in S$ has *finite order* if $\{ x^n \mid n \geq 1 \}$ is finite. Recall that S is *nilpotent* (resp., *nil*) if $S^n = \{0\}$ for some integer $n \geq 1$ (resp., for every $x \in S, x^n = 0$ for some integer $n = n(x) \geq 1$). Thus, a nilpotent semigroup is a nil semigroup, and a finite nil semigroup is a nilpotent semigroup. If every element of S is a zero-divisor (i.e., $Z(S) = S$), then we call S a *zero-divisor semigroup*. Note that we can usually assume that a commutative semigroup S with 0 is a zero-divisor semigroup since $Z(S)$ is an (prime) ideal of S and $G(S) = G(Z(S))$. Clearly, a nonzero nil semigroup, and hence a nonzero nilpotent semigroup, is a zero-divisor semigroup.

A general reference for graph theory is [26], and a general reference for semigroups is [42]. Other definitions will be given as needed.

2 Examples of Zero-Divisor Graphs

Let S be a (multiplicative) commutative semigroup with 0. Associate to S a (simple) graph $G(S)$ with vertices the nonzero zero-divisors of S, and two distinct vertices x and y are adjacent if and only if $xy = 0$. Note that $G(S)$ is the empty graph if and only if $S = \{0\}$ or $Z(S) = \{0\}$ (i.e., $\{0\}$ is a prime semigroup ideal of S). To avoid any trivialities, we will implicitly assume that $G(S)$ is not the empty graph.

In this section, we give several specific examples of "zero-divisor" graphs that have appeared in the literature and show that they are all the zero-divisor graph $G(S)$ for some commutative semigroup S with 0. This illustrates the power of this unifying concept and explains why these "zero-divisor" graphs all share common properties related to diameter and girth.

Example 2.1 Let R be a commutative ring with $1 \neq 0$.

1. The "usual" zero-divisor graph $\Gamma(R)$ defined in [11] has vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. Thus, $\Gamma(R) = G(S)$, where $S = R$ considered as a multiplicative semigroup.
2. Let I be an ideal of R. As in [63], the *ideal-based zero-divisor graph* of R with respect to I is the (simple) graph $\Gamma_I(R)$ with vertices $\{ x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I \}$, and distinct vertices x and y are adjacent if and only if $xy \in I$. Thus, $\Gamma_I(R) = G(S)$, where $S = R/I$ is the Rees semigroup of (the multiplicative semigroup) R with respect to I (i.e., the ideal I collapses to 0). In particular, $\Gamma_{\{0\}}(R) = \Gamma(R)$.
3. Define an (congruence) equivalence relation $\sim$ on R by $x \sim y \Leftrightarrow \mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, and let $R_E = \{ [x] \mid x \in R \}$ be the commutative monoid of (congruence) equivalence classes under the induced multiplication $[x][y] = [xy]$. Note that $[0] = \{0\}$ and $[1] = R \setminus Z(R)$; so $[x] \subseteq Z(R)^*$ for every $x \in R \setminus ([0] \cup [1])$. The *compressed zero-divisor graph* of R is the (simple) graph $\Gamma_E(R)$ with vertices $R_E \setminus \{[0], [1]\}$, and distinct vertices $[x]$ and $[y]$ are adjacent if and only if

$[x][y] = [0]$, if and only if $xy = 0$. Thus, $\Gamma_E(R) = G(R_E)$. This zero-divisor graph was first defined (using different notation) in [57] and has been studied in [8, 9, 29], and [67]. The semigroup analog has been studied in [35] and [38].

4. Let $\sim$ be a multiplicative congruence relation on R (i.e., $x \sim y \Rightarrow xz \sim yz$ for $x, y, z \in R$). As in [10], the *congruence-based zero-divisor graph* of R with respect to $\sim$ is the (simple) graph $\Gamma_\sim(R)$ with vertices $Z(R/\sim) \setminus \{[0]_\sim\}$, and distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if $[xy]_\sim = [0]_\sim$, if and only if $xy \sim 0$. Thus, $\Gamma_\sim(R) = G(R/\sim)$, where $R/\sim = \{\, [x]_\sim \mid x \in R \,\}$ is the commutative monoid of congruence classes under the induced multiplication $[x]_\sim[y]_\sim = [xy]_\sim$. The congruence-based zero-divisor graph includes the three above zero-divisor graphs as special cases.
5. Let S be the semigroup of ideals of R under the usual ideal multiplication. As in [24], $\mathbb{AG}(R) = G(S)$ is called the *annihilating-ideal graph* of R (this zero-divisor graph was first defined in [73]). Similarly, as in [32], define the *annihilating-ideal graph* of a commutative semigroup S with 0 to be $\mathbb{AG}(S) = G(T)$, where T is the semigroup of (semigroup) ideals of S under the usual multiplication of (semigroup) ideals.
6. Let $(S, \wedge)$ be a meet semilattice with least element 0. As in [60], the *zero-divisor graph* of S is the (simple) graph $\Gamma(S)$ with vertices $Z(S) \setminus \{0\} = \{\, 0 \neq x \in S \mid x \wedge y = 0$ for some $0 \neq y \in S \,\}$, and distinct vertices x and y are adjacent if and only if $x \wedge y = 0$. Recall that S becomes a commutative (Boolean) semigroup S' with 0 under the multiplication $xy = x \wedge y$; so $\Gamma(S) = G(S')$. Similar zero-divisor graphs have been defined for posets and lattices [see [39, 40, 46, 48, 53], and Theorem 4.1(1)].

However, not all "zero-divisor" graphs can be realized as $G(S)$ for a suitable commutative semigroup S with 0. For example, as in [19], the *annihilator graph* of a commutative ring R with $1 \neq 0$ is the (simple) graph $AG(R)$ with vertices $Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $\operatorname{ann}_R(x) \cup \operatorname{ann}_R(y) \neq \operatorname{ann}_R(xy)$. Then $\Gamma(R)$ is a subgraph of $AG(R)$, and may be a proper subgraph (e.g., $\Gamma(\mathbb{Z}_8) = K_{1,2}$, while $AG(\mathbb{Z}_8) = K_3$). Thus, $AG(R)$ need not be a $G(S)$. Similarly, as in [1], one can also define the *annihilator graph* $AG(S)$ of a commutative semigroup S with 0. The zero-divisor graph $G(S)$ is a subgraph of $AG(S)$.

As in [81], for a commutative semigroup S with 0, let $\overline{G}(S)$ be the (simple) graph with vertices $Z(S) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xSy = \{0\}$. Then $G(S)$ is a subgraph of $\overline{G}(S)$, and may be a proper subgraph (e.g., if $S = \{0, 2, 4, 6\} \subseteq \mathbb{Z}_8$, then $G(S) = K_{1,2}$, while $\overline{G}(S) = K_3$).

3 Some Properties of the Zero-Divisor Graph $G(S)$

In this section, we give some properties of the zero-divisor graph $G(S)$ of a commutative semigroup S with 0 and are particularly interested in which graphs can be realized as $G(S)$ for some commutative semigroup S with 0. We start with some basic properties of $G(S)$. Parts (1)–(3) of Theorem 3.1 were first proved for $\Gamma(R)$ (cf. [11, 12, 31], and [57]).

Theorem 3.1 *Let S be a commutative semigroup with* 0.

1. ([33, Theorem 1.2]) $G(S)$ *is connected with* $diam(G(S)) \in \{0, 1, 2, 3\}$.
2. ([33, Theorem 1.3]) *If $G(S)$ does not contain a cycle, then $G(S)$ is a connected subgraph of two star graphs whose centers are connected by a single edge.*
3. ([33, Theorem 1.5]) *If $G(S)$ contains a cycle, then the core of $G(S)$ is a union of triangles and squares, and any vertex not in the core of $G(S)$ is an end. In particular,* $gr(G(S)) \in \{3, 4, \infty\}$.
4. ([30, Theorem 1(4)]) *For every pair x, y of distinct nonadjacent vertices of $G(S)$, there is a vertex z of $G(S)$ with* $N(x) \cup N(y) \subseteq \overline{N(z)}$.

Remark 3.2 (1) In Theorem 3.1(4), it is easily shown that $N(x) \cup N(y) \subsetneq \overline{N(z)}$ (for any such z), and either case $z \in N(x) \cup N(y)$ or $z \notin N(x) \cup N(y)$ may occur. Moreover, we can always choose $z = xy$, but there may be other choices for z.

(2) In [53], a (simple) connected graph which satisfies condition (4) of Theorem 3.1 is called a *compact graph*. In [53, Theorem 3.1], it was shown that a simple graph G is the zero-divisor graph of a poset if and only if G is a compact graph.

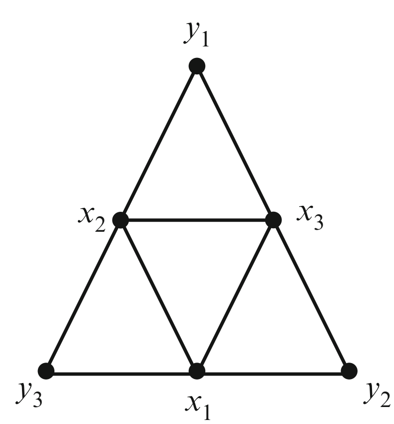
For small graphs, conditions (1), (3), and (4) of Theorem 3.1 actually characterize zero-divisor graphs.

Theorem 3.3 ([30, Theorem 2]) *Let G be a (simple) graph with $|V(G)| \leq 5$ satisfying conditions (1), (3), and (4) of Theorem 3.1. Then $G \cong G(S)$ for some commutative semigroup S with* 0.

([30, Example 2]). In view of Theorem 3.3, Fig. 1 is a graph with six vertices which satisfies conditions (1), (3), and (4) of Theorem 3.1, but G is not the zero-divisor graph of any commutative semigroup with 0. (Also, see [35, Fig. 2, p. 3372].)

The next theorem gives several classes of graphs which can be realized as the zero-divisor graph of a commutative semigroup with 0. As to be expected, many more graphs can be realized as $G(S)$ for a commutative semigroup S with 0 than as

Fig. 1 A graph with six vertices which satisfies conditions (1), (3), and (4) of Theorem 3.1, but G is not the zero-divisor graph of any commutative semigroup with 0



$\Gamma(R)$ for a commutative ring R with $1 \neq 0$ (cf. [11, 12, 64], and [65]). For example, K_n and $K_{1,n}$ (for an integer $n \geq 1$) can be realized as a $G(S)$ for every $n \geq 1$, but can be realized as a $\Gamma(R)$ if and only if $n+1$ is a prime power [11, Theorem 2.10 and p. 439].

Theorem 3.4 ([30, Theorem 3]) *The following graphs are the zero-divisor graph of some commutative semigroup with* 0*.*

1. *A complete graph or a complete graph together with an end.*
2. *A complete bipartite graph or a complete bipartite graph together with an end.*
3. *A refinement of a star graph.*
4. *A graph which has at least one end and diameter* ≤ 2*.*
5. ([33, Theorem 1.3(2)]) *A graph which is the union of two star graphs whose centers are connected by a single edge.*

([30, Example 3]). By (3) and (5) of Theorem 3.4, the refinement of a star graph and the union of two star graphs whose centers are connected by an edge are each the zero-divisor graph of a commutative semigroup with 0. The graph in Fig. 2 is also a refinement of the union of two star graphs with centers at vertex a and vertex b. However, it is not the zero-divisor graph of any commutative semigroup with 0. The vertices a and f do not satisfy condition (4) of Theorem 3.1 since vertex a is adjacent to d and vertex f is adjacent to c, but there is no vertex adjacent to both c and d.

The following theorem gives necessary and sufficient conditions on the semigroup S for $G(S)$ to be a refinement of a star graph (cf. [11, Theorem 2.5] for commutative rings).

Theorem 3.5 ([79, Theorem 1.1]) *Let S be commutative semigroup with* 0 *and $Z(S) \neq \{0\}$. Then $G(S)$ is a refinement of a star graph if and only if either $Z(S)$ is an annihilator ideal (and hence a prime ideal) of S or $Z(S) = A \cup B$, where $A \cong (\mathbb{Z}_2, \cdot)$, $A \cap B = \{0\}$, and A, B are ideals of S.*

For a vertex c of a graph G, let G_c^* be the induced subgraph of G with vertices $V(G_c^*) = V(G) \setminus \{\, u \in V(G) \mid u = c$ or u is an end vertex adjacent to $c\,\}$.

Theorem 3.6 ([79, Theorem 2.3]) *Let S be a set with a commutative binary operation and a zero element* 0 *such that $S = \{0\} \cup \{c\} \cup T \cup S_1$ is the disjoint union of four nonempty subsets. Assume further that $Z(S) = S$, whose zero-divisor*

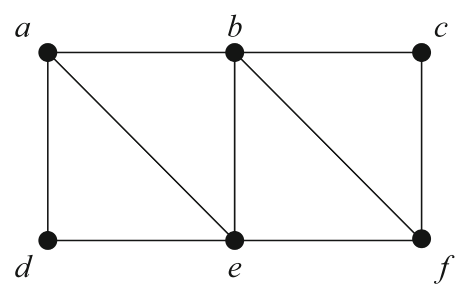


Fig. 2 A graph which is a refinement of the union of two star graphs with centers at vertex a and vertex b. However, it is not the zero-divisor graph of any commutative semigroup with 0

graph $G(S)$ is a refinement of a star graph with center c such that $S_1 = V(G_c^*)$ and G_c^* has at least two components. Then the following statements are equivalent.

1. *S is a commutative zero-divisor semigroup (i.e., the binary operation is associative).*
2. *$S_1^2 = \{0, c\}$, $T^2 \subseteq \{0, c\}$, $c^2 = 0$, and $ts_1 = c$ for every $t \in T$ and $s_1 \in S_1$.*
3. *$S^2 = \{0, c\}$ and $S^3 = \{0\}$.*

Recall that a vertex x of a graph G has *degree m*, denoted by $\deg(x) = m$, if $|N(x)| = m$. For an integer $k \geq 1$, let G_k be the induced subgraph of G with vertices $V(G_k) = \{x \in V(G) \mid \deg(x) \geq k\}$. For a commutative semigroup S with 0 and an integer $k \geq 1$, let $I_k = \{x \in V(G) \mid \deg(x) \geq k\} \cup \{0\}$. Results in the next two theorems from [30] were stated for nilpotent semigroups, but their proofs show that they hold for nil semigroups (i.e, every element is nilpotent).

Theorem 3.7 *Let S be a commutative semigroup with 0.*

1. ([30, Theorem 4]) *I_k is a descending chain of ideals in S.*
2. ([30, Corollary 1]) *The core of $G(S)$ together with $\{0\}$ is an ideal of S whose zero-divisor graph is the core of $G(S)$.*
3. ([30, Corollary 2]) *If S is a nil semigroup, then $G(S)_k = G(I_k)$ for every integer $k \geq 1$.*
4. ([30, Corollary 3]) *Let G be a graph and assume that G_k is not the zero-divisor graph of any commutative semigroup with 0 for some integer $k \geq 1$. Then G is not the zero-divisor graph of any commutative nil semigroup.*
5. ([30, Corollary 4]) *Let G be a graph which is equal to its core, but is not the zero-divisor graph of any commutative semigroup with 0, and let H be the graph obtained from G by adding ends to G. Then H is not the zero-divisor graph of any commutative semigroup with 0.*

Sharper results hold when S is a nil semigroup. A well-known special case is for $\Gamma(R)$ when $Z(R) = \mathrm{nil}(R)$ (e.g., when R is a finite local ring).

Theorem 3.8 *Let S be a commutative semigroup with 0.*

1. ([30, Theorem 5]) *Assume that S is a nil semigroup. Then*
   a. *$diam(G(S)) \in \{0, 1, 2\}$.*
   b. *Every edge in the core of $G(S)$ is the edge of a triangle in $G(S)$. In particular, $gr(G(S)) \in \{3, \infty\}$.*
2. ([30, Corollary 5]) *If every element of S has finite order and some edge in the core of $G(S)$ is the edge of a square, but not a triangle, then S contains a nonzero idempotent element.*

In [35], the authors gave several criteria for a graph G to be a zero-divisor graph in terms of the number of edges of G and adding or removing edges from a given zero-divisor graph $G(S)$. In the next theorem, we are removing edges from K_n (which has $n(n-1)/2$ edges).

Theorem 3.9 ([35, Theorem 2.5(1)]) *Let G be a connected graph with n vertices and $n(n-1)/2 - p$ edges. Then G is the zero-divisor graph of a commutative semigroup with* 0 *if* $0 \leq p \leq \lceil n/2 \rceil + 1$ *(i.e., if G has at least $n(n-1)/2 - \lceil n/2 \rceil - 1$ edges).*

Theorem 3.10 ([35, Theorem 3.22]) *Let $G = G(S)$ be a zero-divisor graph with cycles for a commutative semigroup S with* 0 .

1. *If a is an end adjacent to x in G, then adding another end adjacent to x results in a zero-divisor graph.*
2. *Removing an end from G results in a zero-divisor graph.*

For a commutative semigroup S with 0, let $G^\bullet(S)$ be the (simple) graph with vertices the nonzero zero-divisors of S, and distinct vertices x and y are adjacent if and only if $xy \neq 0$ [in [36], 0 was allowed to be a vertex of $G^\bullet(S)$]. As in [36], a graph G is called *admissible* if $G \cong G^\bullet(S)$ for some commutative zero-divisor semigroup S. In [36], the authors study $G(S)$ by studying $G^\bullet(S)$.

Theorem 3.11 ([36, Theorem 2]) *Given a connected graph G, let G' be the graph obtained by the following procedure: For every edge $a - b$ in G, add a vertex $c_{a,b}$ and edges $a - c_{a,b}$, $b - c_{a,b}$. Then G' is connected and admissible.*

For a graph G, let $\overline{G}$ be the *complement graph* of G (i.e., $V(\overline{G}) = V(G)$ and $a-b$ is an edge in $\overline{G}$ if and only if $a-b$ is not an edge in G for every two distinct vertices a, b of G). Thus, $G^\bullet(S) = \overline{G(S)}$. The next theorem gives some necessary conditions on $\overline{G}$ for G to be admissible.

Theorem 3.12 ([36, Theorem 4], cf. Theorem 3.1) *Let G be an admissible graph.*

1. *$\overline{G}$ has at most one nontrivial component, i.e., with more than one vertex.*
2. *For every connected pair $a, b \in V(\overline{G})$, $d(a,b) \leq 3$.*
3. *The induced cycles in $\overline{G}$ are either* 3*-cycles or* 4*-cycles.*
4. *For every pair a, b of distinct nonadjacent vertices of $\overline{G}$, there is a vertex c of $\overline{G}$ such that $N(a) \cup N(b) \subseteq \overline{N(c)}$.*

Let G be a simple connected graph, and let $S \subseteq V(G)$. Then a vertex x of G is said to *bound* S if for every $y \in N(x)$, we have $\mathrm{d}(y,t) \leq 1$ for every $t \in S$. The set of boundary vertices of S is denoted by $B_G(S)$. A set $S \subseteq V(G)$ is said to be *bounded* if $B_G(S) \neq \emptyset$; otherwise, S is said to be *unbounded* (see [36]).

Theorem 3.13 ([36, Theorem 3 and Corollary (p. 1490)]) *Let G be an admissible graph and $a, b \in V(G)$, not necessarily distinct. Then $ab \in B_G(\{a,b\}) \cup \{0\}$. In particular, if $a - b$ is an edge of G, then $B_G(\{a\}) \neq \emptyset$ and $B_G(\{a,b\}) \neq \emptyset$.*

The following theorem gives some connections between elements in an admissible graph.

Theorem 3.14 *Let G be an admissible graph. Then*

1. ([36, Lemma 1]) *If $a - b$ is an edge of G, then $d(ab, a) \leq 2$ and $d(ab, b) \leq 2$.*
2. ([36, Proposition 5]) *For every $a \in V(G)$, $a^2 \in B_G(\{a\}) \cup \{0\}$.*
3. ([36, Proposition 6]) *If $a - b$ is an edge of G and $a^2 = b^2 = 0$, then a and b are adjacent to a common vertex of G.*
4. ([36, Proposition 7]) *If $a - b$ is an edge of G and $a^2 = 0$, then $ab \notin \overline{N(a)}$.*
5. ([36, Proposition 8]) *If $a - b$ is an edge of G and $a^2 = a$, then $ab \in \overline{N(a)}$.*
6. ([36, Corollary (p. 1495)]) *If $a - b$ is an edge of G such that $a^2 = 0$ and $b^2 = b$, then $ab \in N(b) \setminus N(a)$.*

4 The Number of Zero-Divisor Semigroups

Not only is it of interest to know which graphs can be realized as $G(S)$ for some commutative semigroup S with 0, but more precisely, what are the choices for such semigroups S? The case for commutative semigroups S with 0 and $G(S)$ is somewhat different than for commutative rings R with $1 \neq 0$ and $\Gamma(R)$. It is well known that $|R| \leq |Z(R)|^2$ when $Z(R) \neq \{0\}$; so (up to isomorphism) there are only finitely many commutative rings with $1 \neq 0$ that have a given (nonempty) finite zero-divisor graph. However, for semigroups, one can always adjoin units; so if there is a commutative semigroup S with 0 and $G \cong G(S)$, then for every cardinal number $n \geq |S|$, there is a commutative semigroup $S(n)$ with 0 [and $Z(S(n)) = Z(S)$] such that $G \cong G(S(n))$ and $|S(n)| = n$. Thus, to determine which commutative semigroups with 0 realize a given graph G, we will restrict our attention to commutative zero-divisor semigroups [i.e., $S = Z(S)$].

While it is usually not true that $G(S) \cong G(T)$ implies that $S \cong T$ for commutative semigroups S and T with 0, we can get better results when we restrict to certain classes of zero-divisor semigroups. We first consider the case when S is reduced (i.e., $x^n = 0$ implies $x = 0$). The zero-divisor graph of reduced commutative semigroups with 0 has been studied in [8, 9, 38, 46], and [53]. The next theorem shows that this case reduces to Boolean semigroups (i.e., $x^2 = x$ for every element). Call a monoid S with 0 a *zero-divisor monoid* if $S \setminus \{1\} = Z(S)$. Special cases of the next theorem have been proved in [53, Theorem 4.3] for (1) and [51, Theorem 4.2] for (2).

Theorem 4.1 *1.* ([46, Corollary 1.2]) *The following statements are equivalent for a graph G with at least two vertices.*

a. $G \cong G(S)$ *for some reduced commutative semigroup S with* 0.
b. $G \cong G(S)$ *for some commutative Boolean semigroup S with* 0.
c. $G \cong G(S)$ *for some meet semilattice S.*

2. ([8, Theorem 2.1]) *Let S and T be commutative Boolean zero-divisor monoids. Then $G(S) \cong G(T)$ if and only if $S \cong T$.*

We next give several classes of graphs for which one can determine all possible commutative zero-divisor semigroups with a given graph. However, we will be content to just give the number (up to isomorphism) of such semigroups rather than list them all explicitly.

In [76], the authors gave recursive formulas for the number (up to isomorphism) of commutative zero-divisor semigroups whose zero-divisor graphs are either K_n or $K_n + 1$ (a K_n with an end adjoined) and compute these numbers up to $n = 10$. For example, there are (up to isomorphism) 139 commutative zero-divisor semigroups with zero-divisor graph K_{10} and 7, 101 with zero-divisor graph $K_{10} + 1$. We give the explicit formula for K_n; let $p(m, r)$ be the number of partitions $x_1 + \cdots + x_r = m$ of the integer m with $x_1 \geq x_2 \geq \cdots \geq x_r \geq 1$.

Theorem 4.2 ([76, Theorem 2.2]) *For every integer $n \geq 1$, there are (up to isomorphism)*

$$1 + \sum_{r}^{n} = 1 \sum_{e=0}^{n-r} p(n-e, r)$$

commutative zero-divisor semigroups whose zero-divisor graph is K_n.

We next consider star graphs.

Theorem 4.3 ([71]) *Let $n \geq 1$ be an integer and $f(n)$ be the number (up to isomorphism) of commutative semigroups with n elements. Then there are (up to isomorphism) $n+2+2f(n-1)+2f(n)$ commutative zero-divisor semigroups whose zero-divisor graph is $K_{1,n}$.*

Theorem 4.4 ([79, Theorem 2.13])

1. *If S is a nilpotent commutative semigroup with $G(S)$ a star graph, then $S^4 = \{0\}$.*
2. *For every cardinal number $n \geq 2$, there is a unique (up to isomorphism) nilpotent commutative semigroup $S(n)$ such that $G(S(n)) = K_{1,n}$ and $S(n)^3 \neq \{0\}$.*

Theorem 4.5 *Let n be an integer.*

1. ([79, Theorem 3.6]) *For every $n \geq 2$, there are (up to isomorphism) $n + 2$ nilpotent commutative semigroups with* 0 *whose zero-divisor graph is the star graph K_1, n.*
2. ([69, Theorem 2.1]) *There are (up to isomorphism)* 12 *commutative zero-divisor semigroups whose zero-divisor graph is the star graph $K_1, 2$.*
3. ([69, Theorem 2.2]) *There is (up to isomorphism) a unique commutative zero-divisor semigroup whose zero-divisor graph is the path graph P_4: $a - b - c - d$.*
4. ([69, Theorem 2.5]) *There are (up to isomorphism)* 35 *commutative zero-divisor semigroups whose zero-divisor graph is the graph $K_1, 3$.*
5. ([69, Theorem 2.7]) *There are (up to isomorphism)* 31 *commutative zero-divisor semigroups whose zero-divisor graph is the graph in Fig. 3.*

By [79, p. 339], the number of commutative zero-divisor semigroups whose zero-divisor graph is K_2 (resp., K_3, K_4, and $K_3 + 1$) is 4 (resp., 7, 12, and 22). Combining this with Theorem 4.5 gives all commutative zero-divisor semigroups whose zero-divisor graph has at most four vertices.

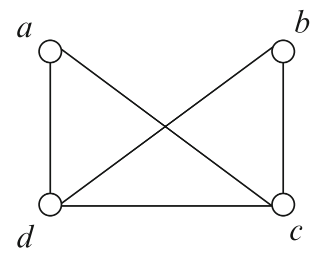


Fig. 3 There are (up to isomorphism) 31 commutative zero-divisor semigroups with this zero-divisor graph

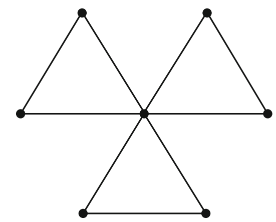


Fig. 4 F_3, i.e., a friendship graph with $|I| = 3$

Recall that a graph G is a *friendship graph* if G is graph-isomorphic to $(\cup_I K_2) + K_1$, for some set I; this graph is denoted by $F_{|I|}$. For example, Fig. 4 is a friendship graph with $|I| = 3$. We call G a *fan-shaped graph* if G is graph-isomorphic to $P_n \cup \{c\}$, where P_n is the path graph on n vertices and c is adjacent to every vertex of P_n, and denote this graph by F'_n.

Theorem 4.6 ([79, Lemma 3.1]) *For every integer $n \geq 2$, there are (up to isomorphism) $\frac{(n+1)(n+2)}{2}$ commutative zero-divisor semigroups whose zero-divisor graph is the friendship graph F_n.*

Theorem 4.7 ([79, Theorem 3.2]) *Let G be the friendship graph F_n together with m end vertices adjacent to its center, where $n \geq 2$, $m \geq 0$. Then there are (up to isomorphism) $\frac{(n+1)(n+2)(m+1)}{2}$ commutative zero-divisor semigroups whose zero-divisor graph is the graph G.*

The number of fan-shaped graphs F'_n for $n \geq 6$ is a special case of the next theorem (let $T = \emptyset$, so the number is $g(n)$). For $n = 2$ (resp., 3, 4, and 5), the number (up to isomorphism) of commutative zero-divisor semigroups whose zero-divisor graph is F'_n is 4 (resp., 12, 47, and 26) (see [68] for $n = 4$ and [80, Theorem 3.1] for $n = 5$).

Theorem 4.8 ([79, Theorem 3.5]) *For every integer $n \geq 6$ and any finite set T, let $G = (P_n \cup T) + c$ be the graph with $G_c^* = P_n$, where P_n is the path graph with n vertices. Then there are (up to isomorphism) $(|T| + 1)g(n)$ commutative zero-divisor semigroups whose zero-divisor graph is the graph G, where* $g(n) = \begin{cases} \frac{1}{2}(2^n + 2^{\frac{n}{2}}) & \text{if } n \text{ is even} \\ \frac{1}{2}(2^n + 2^{\frac{n+1}{2}}) & \text{if } n \text{ is odd.} \end{cases}$

The next two theorems from [77] concern the complete graph K_n with an end adjoined to some vertices of K_n.

Theorem 4.9 *Let n be an integer.*

1. ([77, Theorem 2.1]) *For $n \geq 4$, there is (up to isomorphism) a unique commutative zero-divisor semigroup whose zero-divisor graph is the graph K_n together with two end vertices.*
2. ([77, Theorem 2.2]) *For $n \geq 4$, there is no commutative semigroup with* 0 *whose zero-divisor graph is the graph K_n together with three end vertices.*
3. ([77, Proposition 3.1]) *There are (up to isomorphism)* 20 *commutative zero-divisor semigroups whose zero-divisor is the graph K_3 together with an end vertex.*

Theorem 4.10 ([77, Theorem 3.2]) *For integers n and k with $1 \leq k \leq n$, let $M_{n,k} = K_n \cup \{x_1, \ldots, x_k\}$ be the complete graph K_n with vertices $\{a_1, \ldots, a_n\}$ together with k end vertices $\{x_1, \ldots, x_k\}$, where a_i is adjacent to x_i for every $1 \leq i \leq k$.*

1. *For every integer $n \geq 4$, there is a unique commutative zero-divisor semigroup whose zero-divisor graph is either $M_{3,3}$ or $M_{n,2}$.*
2. ([30, Theorem 3(1)]) *For every integer $n \geq 1$, there are multiple commutative zero-divisor semigroups whose zero-divisor graph is either $M_{n,0}$ (i.e., K_n) or $M_{n,1}$.*
3. *For every integer $n \geq 4$ and $k \geq 3$, there is no commutative zero-divisor semigroup whose zero-divisor graph is $M_{n,k}$.*
4. *There are (up to isomorphism) three commutative zero-divisor semigroups whose zero-divisor graph is $M_{3,2}$.*

For an integer $n \geq 4$, let $T_n(2,2) = K_n \cup \{x_1, x_2\}$ be the complete graph K_n with vertices $M_n = \{a_1, \ldots, a_n\}$ together with the edges: $x_1 - a_1$, $x_1 - a_2$, $x_2 - a_3$, and $x_2 - a_4$. For example, Fig. 5 is the graph $T_4(2,2)$. The following two theorems from [41] give the number (up to isomorphism) of commutative zero-divisor semigroups with zero-divisor graph $T_n(2,2)$ for every integer $n \geq 4$.

Theorem 4.11 1. ([41, Lemma 2.1]) *There is no commutative zero-divisor semigroup whose zero-divisor graph is $T_4(2,2)$.*
2. ([41, Theorem 2.2]) *There are (up to isomorphism)* 18 *commutative zero-divisor semigroups whose zero-divisor graph is $T_5(2,2)$.*

Theorem 4.12 ([41, Theorem 2.3]) *Let $n \geq 6$ be an integer and $M_n(2,2) = \{a_1, \ldots, a_n\} \cup \{0, x_1, x_2\}$. Then $M_n(2,2)$ is a commutative zero-divisor semigroup whose zero-divisor graph is $T_n(2,2)$ if and only if the following conditions hold.*

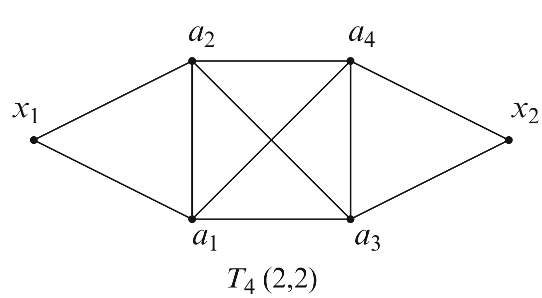


Fig. 5 The graph $T_4(2,2)$

1. $a_i x_1 = 0$ *(*$i = 1, 2$*),* $a_j x_2 = 0$ *(*$j = 3, 4$*),* $x_i^2 = x_i$ *(*$i = 1, 2$*),* $a_i a_j = 0$ *for every* $i \neq j$*,* $a_i^2 \in \{0, a_1, a_2\}$ *(*$i = 1, 2$*), and* $a_j^2 \in \{0, a_3, a_4\}$ *(*$j = 3, 4$*).*

2. $x_1 x_2 \in \{a_5, \ldots, a_n\}$*. If* $x_1 x_2 = a_t$*, then* $a_t x_i = a_t$ *(*$i = 1, 2$*),* $a_t^2 = a_t$ *and* $a_r^2 = 0$ *for every* $r \geq 5$ *and* $r \neq t$*.*

3. $a_r x_1 \in \{a_3, a_4\}$ *for every* $r \neq 1, 2, t$*. If* $a_r x_1 = a_3(a_4)$ *for* $r \neq 3(4)$*, then* $a_3 x_1 = a_3$ *(*$a_4 x_1 = a_4$*) and* $a_2^3 = 0$ *(*$a_2^4 = 0$*)). In particular, if* $a_4 x_1 = a_3$ *(*$a_3 x_1 = a_4$*), then* $a_4^2 = 0$ *(*$a_3^2 = 0$*).*

4. $a_r x_2 \in \{a_1, a_2\}$ *for every* $r \neq 3, 4, t$*. If* $a_r x_2 = a_1(a_2)$ *for* $r \neq 1(2)$*, then* $a_1 x_2 = a_1$ *(*$a_2 x_2 = a_2$*) and* $a_1^2 = 0$ *(*$a_2^2 = 0$*). In particular, if* $a_2 x_1 = a_1$ *(*$a_1 x_2 = a_2$*), then* $a_2^2 = 0$ *(*$a_1^2 = 0$*).*

Moreover, if P_n *is the number (up to isomorphism) of commutative zero-divisor semigroups with zero-divisor graph* $T_n(2, 2)$*, then*

$$P_n = \begin{cases} \frac{1}{48}(n^3 - 6n^2 + 89n + 204) & \text{if } n = 4m + 1 \\ \frac{1}{48}(n^3 + n^2 + 64n - 12) & \text{if } n = 4m + 2 \\ \frac{1}{48}(n^3 - 3n^2 + 71n + 219) & \text{if } n = 4m + 3 \\ \frac{1}{48}(n^3 - 6n^2 + 80n + 144) & \text{if } n = 4m. \end{cases}$$

5 Other Results

We conclude this survey article by referencing a few other results on zero-divisor graphs. Many topics related to associating graphs to algebraic systems have been left untouched; the interested reader may consult the seven survey articles mentioned in the introduction, unreferenced papers in the bibliography, and MathSciNet for many more relevant articles.

Remark 5.1 Some more results.

1. In [27, 62], and [74], the authors studied directed zero-divisor graphs of a noncommutative semigroup with 0.
2. It was shown in [51] that a graph G with more than two vertices has a unique corresponding commutative zero-divisor semigroup if G is a zero-divisor graph of some Boolean ring.
3. In [9], the authors determined the monoids R_E for which $\Gamma_E(R) = G(R_E)$ is a star graph.
4. For other types of graphs associated to semigroups, see, for example, [1, 5, 6, 25, 32], and [81].
5. The authors in [35, 52], and [54] studied commutative zero-divisor semigroups whose zero-divisor graphs are complete r-partite graphs.
6. In [70], the authors determined the number (up to isomorphism) of commutative rings and semigroups whose zero-divisor graphs are regular polyhedra.

7. The authors in [78] studied sub-semigroups determined by the zero-divisor graph.
8. The authors in [15] studied minimal paths in commutating graphs of semigroups.
9. For graphs associated to groups, see, for example, [14, 22], and [50].
10. For graphs of posets, lattices, semilattices, or Boolean monoids, see, for example, [8, 39, 40, 44, 46, 48], and [53].
11. The authors in [27] (resp., [28]) studied the zero-divisor graph (resp., annihilator graph) of near rings.
12. The author in [47] studied the zero-divisor graph of a groupoid.
13. In [45, 66], and [72], the authors gave algorithms for determining if a given graph can be realized as the zero-divisor graph of a commutative ring with $1 \neq 0$.
14. In [33, 40, 44, 53, 54], and [60], the authors studied colorings of commutative semigroups with 0.

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